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# Symmetric tensor rank over an infinite field

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**Abstract** Here we prove two upper bounds (one for bivariate polynomials, one for multivariate ones) for the symmetric tensor rank with respect to an infinite field with characteristic $\neq 2$.

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**الملخص**

نثبت هنا وجود حدين علويين (أحدهما لكثيرات الحدود ثنائية المتغير والآخر لكثيرات الحدود متعددة المتغيرات) لرتبة الموتّر المتناظر بالنسبة لحقل لانهائي مميزه $\neq 2$.

## 1 Introduction

There is a huge and fast growing literature on the rank of tensors and on the symmetric tensor rank of symmetric tensors [4,6,9,10,12,14–16]. There are real world applications and motivations (e.g. signal processing) behind these papers. Most of the papers are over $\mathbb{C}$ (or over an algebraically closed field), but real tensors and real polynomials are also quite studied [13,16]. Here, we work over an arbitrary infinite field and extend to the non-algebraically closed case, a known upper bound for the symmetric tensor rank.

Let $K$ be a field and $\overline{K}$ its algebraic closure. For any $A \subseteq \mathbb{P}^r(K)$ let $\langle A \rangle_K$ denote the $K$-linear span of $A$, i.e., the minimal $K$-linear subspace of $\mathbb{P}^r(K)$ containing $A$. Set $\langle A \rangle := \langle A \rangle_{\overline{K}}$. Notice that $\langle A \rangle_K = \langle A \rangle \cap \mathbb{P}^r(K)$ for any set $A \subseteq \mathbb{P}^r(K)$. Fix an integer $r > 0$ and a set $Y \subset \mathbb{P}^r(K)$. For each $P \in \mathbb{P}^r(K)$ the rank $r_Y(P)$ of $P$ with respect to $Y$ is the minimal cardinality of a set $S \subseteq Y$ such that $P \in \langle S \rangle_K$ (or, equivalently, $P \in \langle S \rangle$), with the convention $r_Y(P) = +\infty$ if $P \notin \langle Y \rangle_K$. Take a projective variety $X \subseteq \mathbb{P}^r$ defined over $K$. We have two sets $X(K) \subseteq \mathbb{P}^r(K)$ and $X(\overline{K}) \subseteq \mathbb{P}^r(\overline{K})$. Obviously $r_{X(K)}(P) \geq r_{X(\overline{K})}(P)$ for any $P \in \mathbb{P}^n(K)$. Quite often the latter inequality is a strict inequality (even when $K = \mathbb{R}$) [13]. Recently we realized that in the case of the symmetric tensor rank (see below for its definition) many known upper bounds for $r_{X(\overline{K})}(P)$ hold over $K$ assuming that $K$ is infinite. We first prove the following result.

**Theorem 1.1** *Fix an integer $d \geq 2$. Let $K$ be a field such that char$(K) \neq 2$ and either $K$ is infinite or $\sharp(K) \geq 2d-2$. Let $C \subset \mathbb{P}^d$ be a degree $d$ linearly normal embedding of $\mathbb{P}^1$ defined over $K$ (equivalently, $C$ is a rational normal curve of $\mathbb{P}^d$ defined over $K$ and with $C(K) \neq \emptyset$). Then*

(a) $r_{C(K)}(P) \leq d$ *for all* $P \in \mathbb{P}^d(K)$.
(b) *For any* $A \subset C(K)$ *such that* $\sharp(A) = d-2$ *there is* $S \subset C(K)$ *such that* $A \subset S$, $\sharp(S) = d$ *and* $P \in \langle S \rangle_K$.

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(c) *Assume that $K$ is infinite. For any finite $B \subset C(K)$ and any $A \subset C(K)$ such that $\sharp(A) = d-2$ and $A \cap B = \emptyset$ there is $S \subset C(K)$ such that $A \subset S$, $B \cap S = \emptyset$, $\sharp(S) = d$ and $P \in \langle S\rangle_K$.*

Theorem 1.1 concerns the one-dimensional case (i.e., the bivariate case) of the following set-up.

Fix an infinite field $K$ and positive integers $m, d$. Set $N := \binom{m+d}{m} - 1$. For any field $L \supseteq K$ let $\nu_{d,L} : \mathbb{P}^m(L) \to \mathbb{P}^N(L)$ denote the order $d$ Veronese embedding of $\mathbb{P}^m(L)$ obtained using the $L$-vector space of all degree $d$ homogeneous polynomials in $m+1$ variables. Set $X_{m,d}(L) := \nu_{d,L}(\mathbb{P}^m(L))$. The set $X_{m,d}(L)$ is the set of $L$-points of the order $d$ Veronese embedding $X_{m,d}$ of $\mathbb{P}^m$. Notice that $\nu_{d,K} = \nu_{d,L}|\mathbb{P}^m(K)$. Since $L$ is infinite, $\mathbb{P}^m(L)$ is Zariski dense in $\mathbb{P}^m(\overline{L})$. Since $L$ is infinite, for every $f \in L[x_0, \dots, x_m]\backslash\{0\}$ there is $(b_0, \dots, b_m) \in L^{m+1}$ such that $f(b_0, \dots, b_m) \neq 0$. Hence the set $X_{m,d}(L)$ spans $\mathbb{P}^N(L)$. Hence for each $P \in \mathbb{P}^N(L)$, $sr_L(P) := r_{X_{m,d}(L)}(P)$ is a positive integer, often called the $L$-symmetric tensor rank of $P$. Now assume $P \in \mathbb{P}^N(K)$. Since $X_{m,d}(L) \supseteq X_{m,d}(K)$, we have $sr_L(P) \le sr_K(P)$. Hence $sr_K(P) \ge sr_{\overline{K}}(P)$. The integer $sr(P) := sr_{m,d,\overline{K}}(P)$ is the usual symmetric tensor rank of $P$ [14]. Using Theorem 1.1, we are able to prove the following extension of Ballico [5, Theorem 1], to a non-algebraically closed field.

**Theorem 1.2** *Fix integers $m, d$ such that $m \ge 5$ and $d \ge 3$. Let $K$ be an infinite field such that $char(K) \neq 2$. For any $P \in \mathbb{P}^N(K)$ we have*

$$sr_K(P) \le d \cdot \lceil \binom{m+d}{m} / (m+1) \rceil.$$

The proof of the case $m \ge 2$ of Theorem 1.2 is only a minor modification of the proof of [5, Theorem 1]. For multivariate polynomials, Białynicki-Birula and Schinzel [7,8] proved a very good upper bound for the symmetric tensor rank. Assume char$(\overline{K}) = 0$. Let $X(\overline{K}) \subset \mathbb{P}^n(\overline{K})$ be an integral and non-degenerate variety defined over $\overline{K}$. Landsberg and Teitler [15, Proposition 5.1] proved that $r_{X(\overline{K})}(P) \le n - \dim(X) + 1$ for all $P \in \mathbb{P}^n(\overline{K})$. This bound is sharp for bivariate polynomials (i.e., if $X(\overline{K})$ a rational normal curve) and for several other curves.

## 2 The proofs

*Proof of Theorem 1.1.* We first check part (b) and hence part (a).

Since $C(K) \neq \emptyset$, either $C(K)$ is infinite (case $K$ infinite) or $\sharp(C(K)) = \sharp(K) + 1$ (case $K$ finite). Fix $P \in \mathbb{P}^d(K)$ and $A \subset C(K)$ such that $\sharp(A) = d-2$. We have $\sharp(C(K)\backslash A) \ge d+1$. Since $C$ is a rational normal curve, we have $\dim(\langle Z\rangle) = \deg(Z) - 1$ for every zero-dimensional scheme $Z \subset C(\overline{K})$ with $\deg(Z) \le d+1$. Hence $\dim(\langle A\rangle_K) = d-3$ and $\langle A\rangle \cap C(\overline{K}) = A$ as schemes. If $P \in \langle A\rangle$, then we are done. Hence we may assume $\dim(\langle \{P\} \cup A\rangle) = d-2$. Notice that $\langle \{P\} \cup A\rangle \cap C(\overline{K})$ has degree $\le d-1$ and it is defined over $K$. Let $\ell : \mathbb{P}^d(\overline{K})\backslash\langle A\rangle_{\overline{K}} \to \mathbb{P}^2(\overline{K})$ denote the linear projection from $\langle A\rangle_{\overline{K}}$. Since $\langle A\rangle \cap C(\overline{K}) = A$, the restriction of $\ell$ induces a morphism $\eta : C(\overline{K})\backslash A \to \mathbb{P}^2(\overline{K})$. Let $Y(\overline{K})$ be the closure of $\eta(C(\overline{K})\backslash A)$ in $\mathbb{P}^2(\overline{K})$. Since $\dim(\langle Z\rangle) = \deg(Z) - 1$ for every zero-dimensional scheme $Z \subset C(\overline{K})$ with $\deg(Z) \le d+1$, $Y(\overline{K})$ is a smooth conic and $\eta$ is injective. Since $\ell$ is defined over $K$, $\eta$ is defined over $K$. For each $Q \in A$ set $A_Q := (A\backslash\{Q\}) \cup 2Q$ seen as a degree $d-1$ effective divisor of $Q$. Since $\dim(\langle Z\rangle) = \deg(Z) - 1$ for every zero-dimensional scheme $Z \subset C(\overline{K})$ with $\deg(Z) \le d$, $Y(\overline{K})\backslash\eta(C(\overline{K})\backslash A)$ is formed by $d-2$ points, each of them defined over $K$ and each of them being one of the points $\ell(\langle A_Q\rangle\backslash\langle A\rangle)$, $Q \in A$. Hence $\ell(P) \notin Y(\overline{K})\backslash\eta(C(\overline{K})\backslash A)$. Since char$(K) \neq 2$, for any smooth conic $E \subset \mathbb{P}^2(\overline{K})$ and any $O \in \mathbb{P}^2(\overline{K})$, there are exactly two tangent lines of $E$ containing $O$. Since char$(K) \neq 2$ and $Y(\overline{K})$ is a smooth conic, there are at most two tangent lines to $Y(\overline{K})$ passing through $\ell(P)$. Since $\eta(C(K)\backslash A)$ has at least cardinality 3, there is a line $T \subset \mathbb{P}^2(\overline{K})$ containing $\ell(P)$, a point $B \in \eta(C(K)\backslash A)$ and not tangent to $Y(\overline{K})$. Since $\ell(P)$ and $B$ are defined over $K$, the line $T$ is defined over $K$. Hence the other point, $B'$, of $T \cap Y(\overline{K})$ is defined over $K$. Notice that $B'$ is uniquely determined by $P$ and $B$. Since $\sharp(\eta(C(K)\backslash A)) \ge d+1$, we may find $B$ with the additional condition that $B' \notin Y(\overline{K})\backslash\eta(C(\overline{K})\backslash A)$. Hence there are $O \in C(K)\backslash A$ and $O' \in C(K)\backslash A$ such that $\ell(O) = B$ and $B' = \ell(O')$. Since $\ell(P) \in T(K) = \langle\{B, B'\}\rangle_K$, we have $P \in \langle A \cup \{O, O'\}\rangle_K$. Take $S := A \cup \{O, O'\}$.

Now assume that $K$ is infinite and take a finite $B \subset C(K)$ such that $B \cap A = \emptyset$. Since $A \cap B = \emptyset$, the set $\ell(B)$ is well defined. Since $\ell(B)$ is finite, we may find $T$ as above with the additional condition $T \cap \ell(B) = \emptyset$. $\square$

*Proof of Theorem 1.2.* Set $\alpha := \lceil \binom{m+d}{m}/(m+1) \rceil$. Fix $P \in \mathbb{P}^N(K)$.

First assume $\mathrm{char}(K) = 0$. For each $U \in X_{n,d}(\overline{K})$ let $T_U X_{m,d}$ be the tangent space of $X_{m,d}(\overline{K})$ at $U$ seen as an $m$-dimensional $\overline{K}$-linear subspace of $\mathbb{P}^N(\overline{K})$. For any $U \in \mathbb{P}^N$ set $T_{U,K} X_{m,d} := T_U X_{m,d} \cap \mathbb{P}^N(K)$. Since the morphism $\nu_{d,\overline{K}}$ is defined over $K$, $T_{U,K} X_{m,d}$ is an $m$-dimensional $K$-linear subspace. Since $m \geq 5$ and $d \geq 3$, a theorem of J. Alexander and A. Hirschowitz [2,3] says that $\alpha$ is the first positive integer $t$ such that $\langle \cup_{Q \in B} T_{\nu_d(Q)} X_{m,d} \rangle = \mathbb{P}^N(\overline{K})$ for a general $B \subset \mathbb{P}^m(\overline{K})$ with cardinality $t$ (Terracini's lemma [1], part 2 of Corollary 1.11). In positive characteristic only one implication holds in Terracini's lemma [1, part 1 of Corollary 1.11]. This is the version of Alexander–Hirschowitz theorem proved in arbitrary characteristic by Chandler [11, Theorem 1]. Hence in arbitrary characteristic $\langle \cup_{Q \in B} T_{\nu_d(Q)} X_{m,d} \rangle = \mathbb{P}^N(\overline{K})$ for a general $B \subset \mathbb{P}^m(\overline{K})$ with cardinality $\alpha$. Since $K$ is infinite, $\mathbb{P}^m(K)$ is Zariski dense in $\mathbb{P}^m(\overline{K})$. Hence there is $B \subset \mathbb{P}^m(K)$ such that $\langle \cup_{Q \in B} T_{\nu_d(Q)} X_{m,d} \rangle = \mathbb{P}^N(\overline{K})$. Hence $\langle \cup_{Q \in B} T_{\nu_d(Q),K} X_{m,d} \rangle = \mathbb{P}^N(K)$. Hence there are $O_b \in T_{\nu_d(B),K} X_{m,d}, b \in B$, such that $P \in \langle \cup_{b \in B} O_b \rangle$. Fix $b \in B$. Let $2b$ be the closed subscheme of $\mathbb{P}^m(\overline{K})$ with $(\mathcal{I}_b)^2$ as its ideal sheaf. The scheme $2b$ is a degree $m+1$ zero-dimensional scheme defined over $K$. Hence it is defined the $K$-linear space $\langle \nu_d(2b) \rangle_K$. Since $\nu_{d,\overline{K}}$ is an embedding, we have $\dim(\langle \nu_d(2b) \rangle_K) = m+1$. We have $\langle \nu_d(2b) \rangle_K = T_{\nu_d(b),K} X_{m,d}$. Let $J \subset \mathbb{P}^N(\overline{K})$ be the line spanned by $\nu_{d,K}(b)$ and by $O_b$. Since $\{\nu_{d,K}(b), O_b\} \subset \mathbb{P}^N(K)$, the line $J$ is defined over $K$. Hence the scheme $Z := \nu_{d,K}(2b) \cap J$ is defined over $K$. We have $\deg(Z) = 2$. Hence there is a degree 2 zero-dimensional scheme $W \subset 2b$ such that $O_b \in \langle \nu_d(W) \rangle$. Notice that $\langle \nu_d(W) \rangle$ is the line $J$ spanned by $\nu_d(b)$ and by $O_b$ and that $\deg(Z) = \deg(W) = 2$. Since $b$ and $O_b$ are defined over $K$, the line $\langle \nu_d(Z_b) \rangle$ is defined over $K$. Hence $\langle \nu_d(W) \rangle_K$ is a well-defined $K$-line. Set $D := \langle W \rangle \subset \mathbb{P}^m(\overline{K})$. $D$ is a line defined over $K$. Hence $C := \nu_{d,\overline{K}}(D)$ is a rational normal curve defined over $K$. Since $b \in B$, we have $C(K) \neq \emptyset$. Since $O_b \in \mathbb{P}^N(K)$, we have $O_b \in \langle \nu_d(W) \rangle \cap \mathbb{P}^N(K) = \langle \nu_d(W) \rangle_K$. Since $O_b \in \langle \nu_d(D_b) \rangle_K$, there is $S_b \subset \mathbb{P}^m(K)$ such that $\sharp(S_b) \leq d$ and $O_b \in \langle \nu_d(S_b) \rangle_K$ (Theorem 1.1; only here we use the assumption $\mathrm{char}(K) \neq 2$). Hence the set $\cup_{b \in B} S_b$ proves the inequality $sr_K(P) \leq d\alpha$. □

**Question 1** Fix an integer $d \geq 2$ and a prime $p$. If $p = 2$, then assume $d \geq 3$. Which is the minimal $p$-power $\alpha(d, p)$ such that Theorem 1.1 is true for the field $\mathbb{F}_q$ for all $p$-powers $q \geq \alpha(d, p)$?

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